QE–Differentiable Manifolds

Semester 1, 2013

[1] (10 points) Explain why the 3 dimensional sphere S^3 cannot be diffeomorphic to S^4, the four dimensional sphere.

[2] (20 points) Let M be a compact smooth manifold of dimension $n \geq 1$. Show that there is no smooth embedding map $f \colon M \to \mathbf{R}^n$.

[3] (20 points) Given an example of a smooth vector field on $\mathbf{R}^2$ that is not complete. Justify your answer.

[4] (30 points) Let $(x_1, \ldots, x_4)$ denote the standard coordinate system for $\mathbf{R}^4$. Let $\alpha = x_2 dx_1 + x_3 dx_3 + dx_4$ and $\beta = 2dx_2 + x_1^2 dx_3 + x_1 dx_4$.

(4.1) Find a differential form γ so that the ideal generated by α, β and γ becomes a differential ideal.

(4.2) What is the integral manifold associated with the differential ideal you found as the answer to (4.1)?

[5] (20 points) Let G be a connected Lie group and let U be an open neighborhood of the identity element e. Then show that

$$G = \bigcup_{n=1}^{\infty} U^n,$$

where U^n is the set of $u_1 \cdot \ldots \cdot u_n$ for $u_1, \ldots, u_n \in U$.