Applied Complex Variables

Spring 2013

1. Suppose that f is analytic (holomorphic) in a disc $D_R(z_0) := \{z: |z - z_0| < R\}$ so that f has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Show that for $n \geq 0,\ 0 < r < R,$

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta})\, e^{-in\theta} d\theta$$

Hint. $a_n n! = f^{(n)}(z_0)$ and apply Cauchy integral formula.

2. Assuming you know

$$\lim_{R\to\infty} \int_{-R}^{R} e^{-\pi x^2} dx = 1,$$

show that

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx.$$

Hint. Consider a contour integral on a rectangle with vertices $R, R + i\xi, -R + i\xi, -R$ and apply Cauchy's theorem.

3. Use the residue theorem to prove that

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2} dx = \pi.$$

Hint. Consider the function $f(z) = \frac{1}{1+z^2}$.