Algebra II Qualifying Examination

All rings here are unital (containing 1) and commutative.

1. Let p be a prime, $q = p^n$ and $\mathbb{F}_q$ be the finite field of q elements. Show that

   (a) $\mathbb{F}_q$ is Galois over $\mathbb{F}_p$;

   [10 points]

   (b) the Frobenius map $x \overset{\sigma}{\mapsto} x^p$ generates the Galois group of $\mathbb{F}_q$ over $\mathbb{F}_p$.

   [10 points]

2. (a) Let A be a ring with a unit, M a finitely generated A-module and I an ideal of A such that $IM = M$. Show that there is a $a \in I$ such that $(1 - a)M = 0$.

   [10 points]

   (b) Let A be an Artinian local ring, $\mathfrak{m}$ its maximal ideal, and $k = A/\mathfrak{m}$ its residual field. Show that if $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq 1$ then every ideal of A is principal.

   [10 points]

3. Let $L : K$ be a finite Galois extension, and $E \subset L$ is a subfield of L containing K.

   (a) Show that for any $\sigma \in \mathrm{Gal}(L/K)$, $\sigma(E)$ is a subfield of L containing K, and

   $$\mathrm{Gal}(L/\sigma(E)) = \sigma \mathrm{Gal}(L/E)\sigma^{-1}.$$

   [10 points]

   (b) Suppose $M \subset L$ is a subfield containing K such that $M : K$ is normal. Show that

   $$\mathrm{Gal}(M/K) = \mathrm{Gal}(L/K)/\mathrm{Gal}(L/M).$$

   [10 points]

4. Fix a unital ring A and $I \subset A$ an ideal. The radical of I, $\sqrt{I}$, is defined as

   $$\{x \in A : x^n \in I \text{ for some } n \in \mathbb{N}\}$$

   (a) Show that $\sqrt{0}$ is the intersection of all prime ideals of A.

   [10 points]

   (b) Let $\mathfrak{q}$ be a primary ideal. Show that $\sqrt{\mathfrak{q}}$ is the smallest prime ideal containing $\mathfrak{q}$.

   [10 points]