# Algebra I Qualifying Exam June, 2013

1. (10 pts) Show that the center of a $p$-group is nontrivial, i.e. has order $> 1$.

2. (a) (10 pts) Give an example of a nonabelian group $G$ of ordert $p^3$.

   (b) (5 pts) For the group $G$ constructed above, compute its center $Z_G$ and order $|Z_G|$.

3. (15 pts) Let $S_n$ denote the symmetric group on $n$ letters, and $V_4$ denote the Klein 4-group. Show that $S_4/V_4 \cong S_3$.

   **Hint**: Construct a short exact sequence of groups $0 \to V_4 \to S_4 \to S_3 \to 0$.

4. (15 pts) Show that the order of a finite field is a power of a prime.

5. (10 pts) Show that there is no infinite ascending chain of ideals in a PID.

6. (15 pts) Let $T$ be a local ring with maximal ideal $\mathcal{M}$. Prove that $\mathcal{M}$ is precisely the set of all non-invertible elements of $T$.

7. Let $R$ be a commutative ring with identity.

   (a) (10 pts) Let $M$ be an $R$-module. Show that $M = 0$ if and only if $M_m = 0$ for all maximal ideals $m$ of $R$.

   (b) (10 pts) Let $x \in R$, and $J$ be a Jacobson radical of $R$. i.e.

   $$J = \text{the intersection of all maximal ideals of R.}$$

   Show that $x \in J$ if and only if $1 - xy$ is a unit in $R$ for every $y \in R$.