Real Analysis Qualifying Exam

December 2012

1. (20 points) Let f be an integrable function on $\mathbb{R}^n$. Given $\varepsilon > 0$, show that there exists a real number $\delta > 0$ such that

$$\int_E |f(x)|\, dx < \varepsilon$$

whenever $E \subset \mathbb{R}^n$ with $|E| < \delta$. (Here, dx denotes Lebesgue measure on $\mathbb{R}^n$, and $|E|$ is Lebesgue measure of E.)

2. (20 points) If f is an integrable function on $\mathbb{R}^n$, show that $\int_{\mathbb{R}^n} |f(a\,x) - f(x)| dx \to 0$ as $a \to 1$. (You may use the following fact without proof: $\int_{\mathbb{R}^n} f(a\,x) dx = a^{-n} \int_{\mathbb{R}^n} f(x) dx$, $a > 0$.)

3. (20 points) Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz transformation, i.e. there is a constant C such that $|T(x) - T(y)| \le C|x - y|$ for all $x, y \in \mathbb{R}^n$. Prove that if E is a measurable set in $\mathbb{R}^n$, then $T(E)$ is also measurable by following the outline:

(a) If H is an F_σ set in $\mathbb{R}^n$ (i.e. H is a countable union of closed sets), then $T(H)$ is also an F_σ set. (Hint. Any closed set can be written as a countable union of compact sets.)

(b) If Z is a set of measure 0 in $\mathbb{R}^n$, then so is $T(Z)$.

(c) If E is a measurable set in $\mathbb{R}^n$, then $T(E)$ is also measurable.

4. (20 points) Suppose that F is a closed set in $\mathbb{R}$ such that $|G| < \infty$, where $G = \mathbb{R} \setminus F$. Let $\delta(x) = \delta(x, F) = \inf\{|x - y| : \; y \in F\}$ for $x \in \mathbb{R}$. Define a function $J(x)$ by

$$J(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x - y|^2} dy, \quad x \in \mathbb{R}.$$

Prove that $J(x) < \infty$ for a.e. $x \in F$. (Suggestion. First show that $\int_F J(x) dx \le A|G|$, and explain why the conclusion follows from this.)

5. (20 points) Assume that the Hardy-Littlewood maximal function f^* on $\mathbb{R}^n$ is of weak type (3, 3), i.e. f^* satisfies

$$|\{x \in \mathbb{R}^n : \; f^*(x) > \alpha\}| \le \left(\frac{A\|f\|_3}{\alpha}\right)^3$$

for some constant A independent of $\alpha > 0$ and $f \in L^3(\mathbb{R}^n)$. Show that

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) dy = f(x) \quad \text{a.e. } x \in \mathbb{R}^n$$

whenever $f \in L^3(\mathbb{R}^n)$. (Here, $B_r(x) = \{y \in \mathbb{R}^n : \; |x - y| < r\}$, $r > 0$, denotes an open ball in $\mathbb{R}^n$.)