Problem 1.(30 points) Let $f(x) = x^7 + 1$ on the interval $[0, 1]$. Let $P(x)$ be the (Lagrange) interpolating polynomial with $P(x_i) = f(x_i)$ for $i = 0, 1, ..., 4$, where the points x_i are given by $0 < x_0 < \cdots < x_4 < 1$. (i) Show that: for any $x_* \in [0, 1]$, there exists a point $\xi \in (0, 1)$ such that

$$f(x_*) - P(x_*) = \frac{f^{(5)}(\xi)}{5!}\omega(x_*)$$

where $\omega(x) = (x - x_0)(x - x_1)\cdots(x - x_4)$. (ii) Show that the Newton divided difference $f[x_0, ..., x_4]$ is given by $f[x_0, ..., x_4] = \frac{f^{(5)}(\xi)}{5!}$ and estimate the error $|f(x_*) - P(x_*)|$.

Problem 2.(30 points) (i) Write down the Peano kernel theorem. (ii) Determine the values c_1, c_2, x_1, x_2 so that the integration formula

$$\int_{-1}^{1} f(x)dx \sim c_1 f(x_1) + c_2 f(x_2)$$

gives the exact integration value when $f(x)$ is a polynomial of degree ≤ 3. Give an example for the formula.

Problem 3.(15points) Let $p_i(x)$, $i = 0, 1, ...,$ be the orthogonal polynomials constructed by Gram-Schmidt orthogonalization and with its leading coefficient 1. Let t_i, $i = 1, ..., n$, be mutually distinct points and let

$$A = \begin{bmatrix} p_0(t_1) & \cdots & p_0(t_n) \\ \vdots & & \vdots \\ p_{n-1}(t_1) & \cdots & p_{n-1}(t_n) \end{bmatrix}.$$

Show that the matrix A is nonsingular.

Problem 4.(25points) Let us consider the two point boundary value problem:

$$\begin{aligned} -u''(x) + u(x) &= f(x) \text{ in } (0, 1), \\ u(0) = 0, u(1) &= 0, \end{aligned} \tag{0.1}$$

where $f(x)$ is a continuous function on $(0, 1)$. (i) Let $0 < x_1 < \cdots < x_n < 1$ be a partition of the interval $(0, 1)$. Set $u_i = u(x_i)$. Using the finite difference method, formulate the problem (0.1) into a matrix problem: $AU = b$ where A is a $n \times n$-matrix, $U = [u_1, \cdots, u_n]$ and $b = [b_1, \cdots, b_n]$. In other words, find A, U and b.

(ii) Give a weak formulation for the problem (0.1) and find A, U and b for the finite element method when the piecewise linear polynomials are used.