

1. Let X be a real-valued random variable with density f (with respect to μ), and ψ be a non-negative Borel-measurable function on \mathfrak{R} . Show that

$$E[\psi(X)] = \int_{\mathfrak{R}} \psi(x)f(x)\mu(dx).$$

2. Let X_1, X_2, \dots be a sequence of random variables converging to X almost surely. Furthermore, there is $M > 0$ such that $|X_n| \leq M$ for all n . Prove that $\lim E|X_n - X| = 0$.

3. Let X and Y be random variables on a probability space (Ω, \mathcal{F}, P) , and $X \leq Y$ almost surely. For any sub σ -field \mathcal{G} of \mathcal{F} , show that

$$E(X|\mathcal{G}) \leq E(Y|\mathcal{G}), \text{ almost surely.}$$

4. Let $\{X_n; n = 0, 1, \dots\}$ be a martingale, and let τ be stopping time such that $0 \leq \tau \leq K$ with probability 1 for some integer K . Verify that $E(X_\tau) = E(X_0)$.

-The end-