2012–2 Qualifying Exam
Differentiable Manifolds and Lie Groups

In what follows, everything is assumed to be smooth (C^∞).

1. Suppose f is a real-valued function on $\mathbb{R}^3$. Define

$$X = \frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y}, \quad Y = \frac{\partial f}{\partial z}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial z}, \quad Z = \frac{\partial f}{\partial z}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial z}.$$

   Define $W(p) = \operatorname{span}\{X(p), Y(p), Z(p)\}$ for $p \in \mathbb{R}^3$.

   (a) (5pts) Show that $W(p)$ has dimension 2 if p is a regular point of f, and 0 otherwise.

   (b) (10pts) Let $N = \{p \in \mathbb{R}^3 \mid p \text{ is a regular point of } f\}$. Show that W is an integrable distribution on N.

   (c) (10pts) Show that f is constant on any connected integral manifold for W on N.

2. (15pts) Show that a 1-form on $\mathbb{R}^3$ is closed if and only if it is exact, without using De Rham theory.

3. Suppose $f, g\colon S^1 \to \mathbb{R}^2$ are embeddings, where S^1 is the unit circle in $\mathbb{R}^2$.

   (a) (10pts) Show that $M := \{(x, y, v) \in S^1 \times S^1 \times \mathbb{R}^2 \mid f(x) - g(y) = v\}$ is a compact submanifold of $S^1 \times S^1 \times \mathbb{R}^2$.

   (b) (15pts) Let $h\colon M \to \mathbb{R}^2$ be the map $h(x, y, v) = v$. Show that v is a regular value of h if and only if $f(S^1)$ meets $g(S^1) + v$ transversely.

4. Suppose M and N are manifolds.

   (a) (10pts) Show that the product $M \times N$ is orientable if M and N are orientable.

   (b) (10pts) Show that if $M \times N$ and M are orientable, then N is orientable.

5. (15pts) Show that a Lie group G of dimension n is orientable and its tangent bundle is diffeomorphic to $G \times \mathbb{R}^n$.