Complex analysis

December 2012

1. (20points) Evaluate the following integrals and justify your answers.

   (a)
   $$\int_0^\infty \frac{x^2}{x^4+1}\,dx$$

   (b)
   $$\int_\gamma \frac{3z+1}{z\,(z-1)^3}\,dz,$$

   where γ is the circle $|z|=2$, oriented counterclockwise.

2. (20points) Evaluate
   $$I=\int_0^\infty \frac{dx}{x^3+a^3},\quad a>0.$$

   Hint: Consider the contour which consists of the line from $(0,0)$ to $(R,0)$, the circular arc from $(R,0)$ to $Re^{2\pi i/3}$ and the line from $Re^{2\pi i/3}$ to $(0,0)$, with $R>0$ large.

3. (10points) Find the first three nonzero terms of the Laurant expansion of the function $f(z)=\tan z$ about $z=\pi/2$.

4. (10points) Show that $e^z-(4z^2+1)=0$ has exactly two roots for $|z|<1$. Hint: Use Rochché's Theorem.

5. (20points) Let $f(z)=\sum a_n z^n$ be an entire function.

   (a) Suppose that $|f(z)|\leq A|z|^N+B$ for all $z\in\mathbb{C}$, where A,B are finite constants and N is a positive integer. Show that f is a polynomial of degree less than or equal to N.

   (b) Suppose that $|f(z_n)|\to\infty$ whenever $|z_n|\to\infty$. Show that f is a polynomial.

6. (20points) Let $u(z)$ be a real-valued harmonic function in the whole complex planc $\mathbb{C}$. Show that if $u(z)\geq 0$ for all $z\in\mathbb{C}$, then u must be constant.