GRADUATION EXAM, ANALYSIS, NOVEMBER 2012

1. (a) Define $f(x) = \log x$ by $\log x = \int_1^x \frac{1}{t} dt$ for $0 < x < \infty$. If $0 < a < \infty$, show that $\log x$ is uniformly continuous on $[a, \infty)$.

(b) Determine whether or not $f(x) = \log x$ is uniformly continuous on $(0, \infty)$. Prove it if it is. Give a counterexample if it is not.

2. (a) Prove the summation by parts formula: $\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} A_n(b_n - b_{n+1}) + A_q b_{q+1} - A_{p-1} b_p$, for $q \geq p \geq 1$. Here, $A_0 = 0$ and $A_n = \sum_{k=1}^{n} a_k$, $n \geq 1$.

(b) Show that $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges if $z \neq 1$ is a complex number such that $|z| = 1$.

3. Given a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$, define $\sigma_n = (a_1 + \cdots + a_n)/n$.

(a) If $a_n \to 2$, show that $\sigma_n \to 2$.

(b) If $\sigma_n \to 2$, does it follow that $a_n \to 2$? Prove your assertions. (If yes, prove it. If no, give a counterexample.)