Q.E. OF ALGEBRA II, DEC. 2012

(1) (25 points) Let K be a finite field extension of a field F.

(a) (25 points) Show that there exist only finitely many subfields of K containing F if and only if there is an $\alpha \in K$ such that $K = F(\alpha)$.

(b) (5 points - bonus points) Find an element $\alpha \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$ such that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(2) (25 points) Let K be a Galois extension over F with a cyclic Galois group of order n generated by σ. Then *additive Hilbert's theorem 90* says that if $\alpha \in K$ satisfies that $\mathrm{Tr}_{K/F}(\alpha) = 0$, then α is of the form $\alpha = \beta - \sigma\beta$ for some $\beta \in K$ (you do not have to prove this).

Let F be a field of characteristic p and let K be a cyclic extension of F of degree p. Prove that $K = F(\alpha)$ where α is a root of the polynomial $x^p - x + a$ for some $a \in F$ (Hint: use the above *additive Hilbert's theorem 90*).

(3) (25 points) Determine the Galois group G of the splitting field of $x^8 - 2$. Use a group presentation of G, i.e. find generators of the Galois group and relations among them, in order to determine G and also find the order of G.

(4) (25 points) Let I be an ideal of a commutative ring R with identity 1. The radical of I, denoted $\mathrm{Rad}(I)$, is defined as the ideal $\bigcap_P P$ where the intersection is taken over all prime ideals containing I.

(a) (5 points) Find $\mathrm{Rad}(6\mathbb{Z})$ and $\mathrm{Rad}(64\mathbb{Z})$ in $\mathbb{Z}$.

(b) (20 points) Prove that $\mathrm{Rad}(I) = \{r \in R \ : \ r^n \in I \quad \text{for some } n > 0\}$.