Algebra I

December 2012

(1) (16pt.) Find all groups of order 35.

(2) (18pt.) Suppose that a group G transitively acts on the right on a set E. Let H be the isotropy subgroup of G corresponding to a point of E. Show that the group of automorphisms of the set E (bijective maps from E to itself that preserve the group action) is isomorphic to $N[H]/H$, where $N[H]$ is the normalizer of H in G.

(3) (17pt.) For positive integers m and n, find the orders of $\mathbb{Z}$-modules $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ and $\mathbb{Z}/n\mathbb{Z} \bigotimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$.

(4) (15pt.) Show that every proper ideal of a commutative ring with 1 is contained in a maximal ideal of the ring.

(5) (17pt.) Let R be an integral domain. Show that an R-module M is torsion-free if and only if $M_{\mathfrak{P}}$ is a torsion-free $R_{\mathfrak{P}}$-module for every prime ideal $\mathfrak{P}$ of the ring R.

(6) (17pt.) Find all two-sided ideals of the ring of 2×2 matrices over $\mathbb{Z}$.