QUALIFYING EXAM, REAL ANALYSIS-2012

1. Prove that for any $f \in L^p(\Omega)$ with $p \in [1, \infty)$, there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset C_0^{\infty}(\Omega)$ such that $\lim_{n\to\infty} \|f - f_n\|_p = 0$. Show that the fact mentioned above does not hold for $p = \infty$.

2. Prove that for any $f \in L^1(\mathbb{R}^n)$, its Fourier transform $\widehat{f}$ is continuous and $\lim_{|x|\to+\infty} \widehat{f}(x) = 0$, that is, $f \in C_0(\mathbb{R}^n)$. Show that the Fourier transform $f \in L^1(\mathbb{R}^n) \to \widehat{f} \in C_0(\mathbb{R}^n)$ is not surjective.

3. Let $f : [0,1] \times [0,1] \to \mathbf{R}^2$ be given by $f(x,y) = (x^2 - y^2, 2xy)$. Find the area of the $f([0,1] \times [0,1])$.

4. Suppose $|f_n| \le g \in L^1$ and $f_n \to f$ in measure, then $\int f \le \lim_{n\to\infty} \int f_n$ and $f_n \to f$ in L^1.

5. Show that every convex function from $\mathbb{R} \to \mathbb{R}$ is continuous.