Numerical Analysis: Written Exam for Ph.D. **June 8, 2012**

Part I.

Problem 1.(20 points) Let $f : [-5, 5] \mapsto (-\infty, \infty)$ be an eleven times differentiable function. Let $h = 1$ be the meshsize. Let $x_i = -5 + ih$ be the nodal point and $f(x_i) = i$ for $i = 0, 1, \cdots 10$. (i) Construct the Lagrange polynomial $P(x)$ which interpolates the function $f(x)$ at the nodal points $x_i, i = 0, \cdots, 10$. (ii) State the error formula and show the derivation.

Problem 2.(20 points) Let $f(x)$ be a real-valued continuous function on the interval $[-1, 1]$. Consider the following quadrature rule

$$\int_{-1}^{1} f(x)dx \sim \frac{1}{3}[f(-1) + 4f(0) + f(1)]. \tag{1}$$

(a) Derive the above quadrature rule. (b) What is the degree of the polynomials to be integrated exactly ? (c) Derive the composite quadrature rule for (1) by partitioning the interval $[-1, 1]$ and state the error formula.

Problem 3.(20 points) Let $\|\mathbf{x}\|_\infty = \max_{1\le k\le n} |x_k|$ for $\mathbf{x} = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ for any integer $n \ge 2$. Let $A = (a_{ij})_{1\le i,j\le n}$ be an $n \times n$-matrix. (a) Show that

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

(b) Let A be an invertible matrix. Let $\mathbf{x}$ be the exact solution of $A\mathbf{x} = \mathbf{b}$ for a given vector $\mathbf{b} \in \mathbb{R}^n$. Let $\Delta\mathbf{x}$ and $\Delta\mathbf{b}$ be the increments of $\mathbf{x}$ and $\mathbf{b}$, respectively. Show that

$$\|\Delta\mathbf{x}\|/\|\mathbf{x}\| \le \text{Cond}(A)\|\Delta\mathbf{b}\|/\|\mathbf{b}\|$$

where $\text{Cond}(A) = \|A^{-1}\|\,\|A\|$.

Problem 4.(20 points) Let $A = (a_{ij})_{1\le i,j\le n}$ be a Hermitian matrix, that is, $A = A^H$. Let the eigenvalues of A be ordered by $\lambda_1 \ge \lambda_2 \ge \cdots \le \lambda_n$. Show the followings:

$$\lambda_1 = \max_{0\ne\mathbf{x}\in\mathbf{C}^n} \frac{\mathbf{x}^H A\mathbf{x}}{\mathbf{x}^H\mathbf{x}}, \quad \lambda_n = \min_{0\ne\mathbf{x}\in\mathbf{C}^n} \frac{\mathbf{x}^H A\mathbf{x}}{\mathbf{x}^H\mathbf{x}}. \tag{2}$$

(Hint: Use the Shur normal form of the matrix).

Problem 5.(20 points) State the QR method for the computation of the eigenvalues of an $n \times n$ matrix A and give the properties.