2012 Spring Qualifying Exam
Algebraic Topology

1. (15pts) Suppose X is a CW complex. Define the cellular cochain complex $C_*(X) = C_*^{CW}(X)$. Describe the generators of $C_n(X)$ and the boundary map. Prove that $H_n(C_*(X))$ is isomorphic to the singular homology $H_n(X)$ when X is finite.

2. (a) (5pts) Describe the orientable surface Σ_g of genus g as a 2-dimensional CW complex with one 0-cell, $2g$ 1-cells, and one 2-cell.

   (b) (10pts) Compute the cellular homology groups of Σ_g.

3. Suppose X is a finite CW complex.

   (a) (5pts) Show that $H_n(X)$ is a finitely generated abelian group for any n.

   (b) (10pts) For any fixed n, show that $H_n(X)$ is finite if and only if $H_n(X;\mathbb{Q}) = 0$.

   (c) (10pts) An abelian group is torsion-free if the identity is the only element of finite order. For any fixed n, show that $H_n(X)$ is torsion-free if $H_n(X;\mathbb{Z}_p) = 0$ for all prime p. Is the converse true? Prove it or give a counterexample.

4. (a) (15pts) Show that if $f\colon S^n \to S^n$ satisfies $f(x) = f(-x)$ for any x, then f has even degree.

   (b) (10pts) Show that a map $f\colon S^n \to S^n$ with odd degree sends some pair of antipodal points to a pair of antipodal points.

5. (20pts) Let $L\colon \{1,2\}\times S^1 \times D^2 \to S^3$ be an embedding. Let $X = S^3 - L(\{1,2\}\times S^1 \times 0)$. Compute the homology groups $H_*(X)$. (Hint: you may use Mayer-Vietoris.)