Complex Analysis Qualifying Exam

Spring 2012

Let $D = \{z \in \mathbb{C} : |z| < 1\}$.

1. (16 points) Use the theory of residues to calculate the integral $\int_0^\infty \frac{1}{1+x^3}\,dx$.

2. (16 points) Let f and g be holomorphic functions in an open set G containing $\overline{D}$. Assume that f has zeros at $z_1, \cdots, z_k \in D$ and no other zeros in G. Let C be the circle $|z| = 1$, traversed once counterclockwise. Calculate $\oint_C \frac{f'(z)}{f(z)}\, g(z)\, dz$.

3. (17 points) Suppose that f is continuous in $\mathbb{C}$ and holomorphic in $\mathbb{C} \setminus I$, where $I = [0,1]$ is a line segment. Show that f is an entire function. (Suggestion: consider Morera's theorem.)

4. (17 points) Suppose $f(z)$ is holomorphic in an open set G containing $\overline{D}$ and that $|f(z)| < 1$ on the circle $|z| = 1$. Show that there is exactly one point $a \in D$ such that $f(a) = a$.

5. (17 points) Suppose $f : D \to \mathbb{C}$ is a nonconstant holomorphic function such that $\operatorname{Im} f(z) \geq 0$ for all $z \in D$. (a) Explain why $\operatorname{Im} f(z) > 0$ for all $z \in D$. (b) If $f(0) = 1 + i$, then show that

$$|f(z)| \leq \sqrt{2}\frac{1+|z|}{1-|z|}, \quad \forall z \in D.$$

6. (17 points) Let $u_1, u_2, \cdots$ be a sequence of harmonic functions in an open set G such that u_k converges uniformly on compact subsets of G. Show that the limit function u is harmonic in G. (Suggestion: consider the Poisson integral.)