Q.E of Algebra I, 201206

1. (40 pt) (1) (20 pt) For a nonnegative integer k, let D_k be the multiplicative group generated by the complex matrices x and y

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, y = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$$

where $\xi = e^{\frac{2\pi i}{k}}$. List the conjugacy classes of D_k.

(2) (20 pt)Let G be a noncyclic group which is generated by two elements a and b each of order 2. Then show that G has a normal cyclic subgroup C of index 2 in G. Also show that if G is finite and of order $2k$, then G is isomorphic to D_k given in the above problem (1).

2. (20 pt) Let R be a commutative ring with identity and P a prime ideal of R. Further let $R_P = S^{-1}R$ be the localization of R at P, where $S = R - P$. Show that the ideal $P_P = S^{-1}P$ in R_P is the unique maximal ideal of R_P.

3. (20 pt) Let G be a finite group and n be a positive integer relatively prime to the order of G.

   (a) (10 pt)Show that if $b^n = c^n$ for two elements b and c in G, then $b = c$.

   (b) (10 pt) For each $x \in G$, there is a unique $y \in G$ such that $y^n = x$.

4. (20pt) Let R be a ring with a center Z. A derivation $D : R \to R$ is a map satisfying $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + D(a)b$ for all $a, b \in R$.

   (a) (10 pt) If D is a derivation of R, prove that $D(Z) \subset Z$.

   (b) (10 pt) If D is a derivation of R and $w \in Z$ is an idempotent, prove that $D(w) = 0$.