GRADUATION EXAM, ANALYSIS, MAY 2012

Do three problems. (Mark the problems you have chosen.)

1. Explain why $f(x) = x^{1/3}$ is continuous at each $0 \leq x \leq 2$. Then determine whether or not f is uniformly continuous on $[0, 2]$. Justify your assertions.

2. Show that $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ converges. (Suggestion: one way is to use the summation by parts formula: $\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} A_n(b_n - b_{n+1}) + A_q b_{q+1} - A_{p-1} b_p$, $q \geq p \geq 1$.)

3. Let f_n be a sequence of real-valued, Riemann-integrable functions on $[0, 3]$. Suppose that $f_n \to f$ uniformly on $[0, 3]$ (for some function f on $[0, 3]$). Show that f is Riemann-integrable on $[0, 3]$ and that

$$\lim_{n\to\infty} \int_0^3 f_n(x)dx = \int_0^3 f(x)dx.$$

4. Suppose that f is a continuous real-valued function on $[1, 3]$ such that $\int_1^3 f(x)x^n \, dx = 0$ for all integers $n \geq 0$. Show that $f(x) = 0$ for all $1 \leq x \leq 3$. (Suggestion. Use the Weierstrass approximation theorem to show that $\int_1^3 f(x)^2 \, dx = 0$. Then show that this implies that $f(x) = 0$ for all x.)